$$\mathcal{S}(\Gamma) = \bigoplus_k \mathcal{S}_k(\Gamma)$$

$$\mathcal{M}(\Gamma) = \bigoplus_k \mathcal{M}_k(\Gamma)$$

Our next goal is to study the structure of these spaces in more detail.

We ~~start~~ by constructing a special cusp form

Recall $E_4(\tau) = 1 + 240q + \dots$

$E_6(\tau) = 1 - 504q + \dots$

Hence $\dfrac{E_4^3 - E_6^2}{1728} = q + 24q^2 + 252q^3 + \dots$

has no constant term and is a modular form of wt 12.

Defn $\Delta := \dfrac{E_4^3 - E_6^2}{1728}$ is called the discriminant function and is a cusp form of wt 12.

We'll see shortly that it is the cusp form of smallest weight for $SL_2(\mathbb{Z})$. ie $\mathcal{S}_k(\Gamma) = \emptyset$ if $k < 12$

§3. The valence formula and the structure of $\mathcal{M}_k$.

Before we study the structure of $\mathcal{M}_k(\Gamma)$ more closely, we start with a generalization of modular forms.

Defn A meromorphic modular function of wt $k \in \mathbb{Z}$ for $\Gamma = SL_2(\mathbb{Z})$ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ s.t.

1) $f(\gamma\tau) = (c\tau+d)^k f(\tau) \quad \forall \gamma \in \Gamma$

2) $\exists\, c > 0$ s.t. f has no singularities in the domain $\operatorname{Im}\tau > c$.

3) f has at most a pole of finite order at $i\infty$.

i.e. $\tilde{f}(q) = \sum_{n=n_0}^{\infty} a_n q^n$ for some $n_0 \in \mathbb{Z}$.
(i.e. f has non-essential sing. at $i\infty$)

We then have the following simple Lemma.

Lemma 3.1 A meromorphic modular function $f \neq 0$ has only finitely many zeroes and poles in $\mathbb{H}$ modulo the action of Γ.

Proof. By hypothesis, $\exists$ a constant C, such that f has no poles in the region $\operatorname{Im}\tau > C$.

After enlarging the region $\operatorname{Im}\tau > C$ if necessary we can assume the same region does not contain any zeroes of f. This is due the fact that the zeroes of an analytic function cannot accumulate to a non-essential singularity when the function $\not\equiv 0$ in a nbhd of this singularity.

Now consider the truncated fundamental domain $\mathcal{F}_C := \{ z \in \mathcal{F} \mid \operatorname{Im} z \leq C \}$

$\mathcal{F}_C$ is compact and hence contain only finitely many zeroes and poles of f.

And these give a finite system of reps mod Γ!

Defn ① Let f be a non-zero meromorphic function on $\mathbb{H}$. $f : \mathbb{H} \to \mathbb{C}$. Let $0 \neq P \in \mathbb{H}$

The unique integer n such that

$\frac{f(z)}{(z-P)^n}$ is holomorphic and non-zero is called the order of f at P

and is denoted by $n_P(f)$ or $\mathrm{ord}_P f$ or $\nu_P(f)$.

If P is a pole of f then $n_P(f) < 0$

If P " " zero of f then $n_P(f) > 0$.

If P is a regular point then $n_P(f) = 0$.

If f has a Laurent expansion at P of the form $f(z) = \sum_{n \geq n_0} a_n (z-P)^n$ then

$n_P(f) = n_0$.

Recall a_{-1} is called the residue of f at P.

(2) We define the order of a meromorphic modular function f at $i\infty$ as n_0 if $\tilde{f}(q) = f(\tau) = \sum_{n \geq n_0} a(n) q^n$

ie it is the order of $\tilde{f}$ at $q = 0$.

Rk If f is weakly modular for Γ then $n_p(f)$ depends only on the image of p in $\Gamma \backslash \mathbb{H}$ since

$f(\gamma\tau) = (c\tau + d)^k f(\tau)$ and $c\tau + d$ has no zeroes or poles in $\mathbb{H}$

(3) The order of a meromorphic function f at a rational point $s = \frac{a}{c} \in \mathbb{Q}$ is defined as the order of

$$f|_\gamma(\tau) = (c\tau + d)^{-k} f(\gamma\tau) \quad \text{at } i\infty.$$

Here $\gamma \in \Gamma$ is any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t $\gamma(\infty) = a/c$.

Note for f modular function for $\Gamma = SL_2(\mathbb{Z})$

$$n_{a/c}(f) = n_\infty(f)$$

We also quickly recall from function theory

<u>Lemma 3.2</u> If f is meromorphic around a point P then

$$\mathrm{Res}_P (f'/f) = n_P(f)$$

<u>Thm 3.3</u> 1) (Cauchy's Formula) Let f be holomorphic on an open subset $U \subseteq \mathbb{C}$ and C a closed contour in U with $P \in U \setminus C$ we have

$$\int_C \frac{f(z)}{z-P}\,dz = 2\pi i\, f(P)\, \mathrm{ind}_C(P)$$

where C is +vely oriented i.e. acw.

2) If $C_r(\phi)$ is an arc of a circle of radius r and angle ϕ around a point P then

$$\lim_{r\to 0} \int_{C_r(\phi)} \frac{f(z)}{z-P}\,dz = i\phi\, f(P)$$

3) (Argument Principle) If f is meromorphic function on an open sset $U \subseteq \mathbb{C}$ and C is closed contour avoiding zeroes and poles of f then

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i \sum_{P \in \mathrm{int}(C)} n_P(f)$$

4) If f is merom. and $C_r(\phi)$ as in 2) then

$$\lim_{r\to 0} \int_{C_r(\phi)} \frac{f'(z)}{f(z)}\,dz = \phi i\, n_P(f)$$

We will apply the argument principle to the boundary of a truncated fund. domain $\bar{F}_T$ to count the zeroes and poles of a merom. mod func.

<u>Thm 3-4</u> (<u>Valence formula</u> or $\frac{k}{12}$ formula)

Let $0 \not\equiv f$ be a meromorphic modular function of wt k. Then

$$\left(\sum_{p \in \Gamma \backslash H} \frac{1}{e_p} n_p(f) \right) + n_\infty(f) = \frac{k}{12}$$

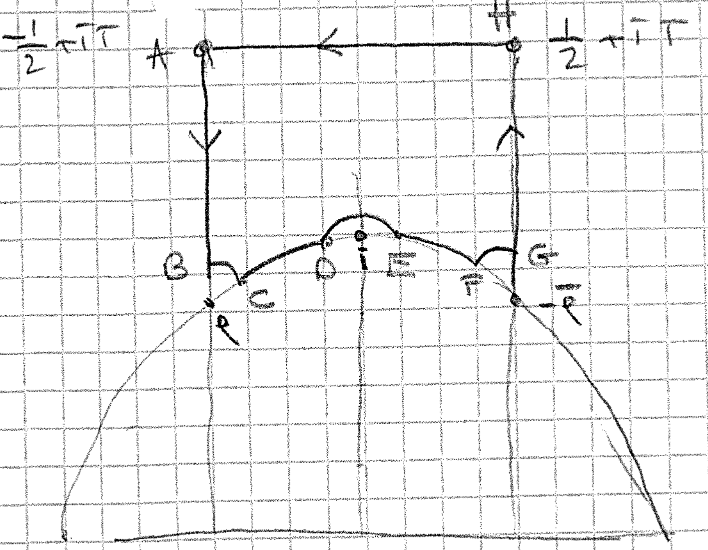
where $e_p = \begin{cases} 3 & \text{if } p \sim_\Gamma e^{2\pi i/3} = \rho \\ 2 & \text{if } p \sim_\Gamma i \\ 1 & \text{otherwise.} \end{cases}$

<u>Proof</u> Choose T large enough so that f has no poles or zeroes for $\operatorname{Im}\tau > T$

(T exists by Lemma 3-1)

Let $F_T = \{\tau \in F \mid \operatorname{Im}\tau < T\}$ the truncated fund. domain and consider the contour $\mathcal{C}$ given by the boundary of F_T.

(We'll first assume f has no poles or zeroes on the boundary of F_T except possibly at $i, \rho, -\bar{\rho}$.



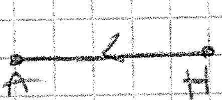
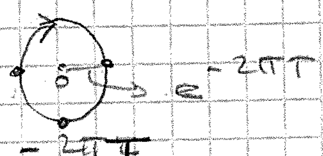
Interior of the contour $\mathcal{C}$ contains a representative of each zero or pole of f not congruent mod Γ to i, ρ, $-\bar{\rho}$.

By Argument principle

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f'}{f}\,d\tau = \sum_{\substack{p \in \Gamma\backslash H \\ p \neq i,\, \rho,\, -\bar{\rho}}} n_p(f)$$

Next we integrate along $\mathcal{C}$ piece by piece

(1) Integrals along AB and GH cancel since $f(\tau+1) = f(\tau)$ and the paths go in opposite directions.

(2) The map $\tau \to e^{2\pi i \tau}$ maps the horizontal line  to  the circle around 0 of radius $e^{-2\pi T}$

since $e^{2\pi i(x+iT)} = e^{2\pi i x}\cdot e^{-2\pi T}$ $\quad -\frac{1}{2} \le x \le 1/2$

$$\tilde{f}(q) = \sum a_n q^n = f(\tau).$$

Then $f'(\tau) = \dfrac{d\tilde{f}(q)}{dq}\cdot\dfrac{dq}{d\tau}$

$$\frac{1}{2\pi i}\int_{HA} \frac{f'(\tau)}{f(\tau)}\,d\tau = \oint_{C_{-2\pi T}(0)} \frac{d\tilde{f}/dq}{\tilde{f}(q)}\,dq$$

$= -n_\infty(f)$ since the circle is traced clockwise.

(3) The integral of f'/f along a circle which contains BC (oriented cw) has the value $-n_\rho(f)$. When the radius ε of this circle tends to zero, the angle $\measuredangle BC$ tends to $2\pi/6$. Hence

$$\frac{1}{2\pi i}\int_{BC} \frac{f'}{f}\,d\tau \xrightarrow{\varepsilon\to 0} \frac{-n_\rho(f)}{6}.$$

Similarly for FG we obtain $\dfrac{-n_{\bar\rho}(f)}{6} = \dfrac{-n_\rho(f)}{6}$.

Similarly

(4) $$\frac{1}{2\pi i}\int_{DE} \frac{f'}{f}\,d\tau \xrightarrow[\varepsilon\to 0]{} \frac{-n_i(f)}{2}.$$

(5) For the remaining arcs CD, $\overline{EF}$

we note that the map $S: \tau \longrightarrow -1/\tau$

take CD to $\overline{FE}$ ie as τ goes from

C to D, $S\tau$ goes from F to E

On the other hand $f(S\tau) = f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$

Hence $\frac{1}{2\pi i}\left[\int_{CD} \frac{f'(\tau)}{f(\tau)} d\tau + \int_{EF} \frac{f'(\tau)}{f(\tau)} d\tau\right]$

$$= \frac{1}{2\pi i}\left[\int_{CD} \frac{f'}{f} d\tau - \frac{f'(S\tau)}{f(S\tau)} d(S\tau)\right]$$

$\frac{d}{d\tau}\left(f(S\tau)\right) = \frac{d}{d\tau}\left(\tau^k f(\tau)\right)$, divide by $f(S\tau)$ on the left & $\tau^k f(\tau)$ on the right

$$\frac{f'(S\tau)\frac{d(S\tau)}{d\tau}}{f(S\tau)} = \frac{\tau^k f'(\tau) + k\tau^{k-1} f(\tau)}{\tau^k f(\tau)}$$

$$\frac{f'(S\tau)}{f(S\tau)} d(S\tau) = \frac{f'(\tau)}{f(\tau)} + k\frac{d\tau}{\tau}$$

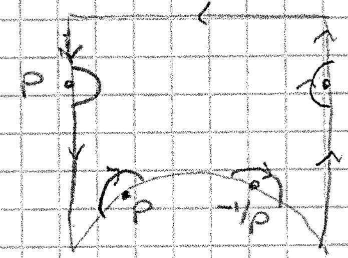
Hence $\frac{1}{2\pi i}\int_{CD+EF} \frac{f'}{f} d\tau = \frac{-k}{2\pi i}\int_{CD} \frac{d\tau}{\tau} \xrightarrow[\tau = e^{2\pi i\theta}]{\varepsilon \to 0} -k\int_{1/3}^{1/4} d\theta = \frac{k}{12}$

Since the angle of C is $2\pi/3$ and angle at $D = \pi/2 = \frac{2\pi}{4}$

Putting everything together gives the thm in the case that there is no zero or pole on the boundary.

If there is a zero on the boundary we similarly go around them with small circles and let $\varepsilon \to 0$ to pick up

$$-\frac{1}{2} n_p(f) - \frac{1}{2} n_p(f) = -n_p(f)$$



As a corollary of the valence formula we can determine the structure of $M_k(\Gamma)$

More precisely

Thm 3.5 Let $k \in 2\mathbb{Z}$. Then

1. $M_0(\Gamma) = \mathbb{C}$
2. $M_k(\Gamma) = 0$ if $k < 0$ or $k = 2$
3. $\dim M_k = 1$ for $k = 4, 6, 8, 10, 14$ and in these cases $M_k = \mathbb{C} E_k$
4. $S_k(\Gamma) = 0$ if $k < 12$ or $k = 14$
   $S_{12}(\Gamma) = \mathbb{C}\Delta$
   $S_k(\Gamma) = \Delta M_{k-12}$ if $k > 14$
5. $M_k(\Gamma) = S_k(\Gamma) \oplus \mathbb{C} E_k$, $k > 2$.

Proof Let $f \in M_k(\Gamma)$. Since f has no poles we have $n_p(f) \geq 0$ $\forall p \in {}_{\Gamma}\backslash H$ and $p = i\infty$ and the valence formula reads

$$\boxed{n_\infty(f) + \frac{1}{2} n_i(f) + \frac{1}{3} n_\rho(f) + \sum_{p \neq i, \rho, \infty} n_p(f) = \frac{k}{12}} \quad (*)$$

(1) Let c be any value taken by $f \in M_0(\Gamma)$. Then $f(\tau) - c \in M_0(\Gamma)$ has a zero but then one of the terms on the left of $(*)$ is strictly positive, on the right we have 0. Hence $f(\tau) - c \equiv 0$ and f is constant

(2). If $k < 0$ then $*$ has no solution in non-negative integers since RHS $= \frac{k}{12} < 0$

Similarly for $k = 2$

$$n_\infty + \frac{1}{2} n_i + \frac{1}{3} n_\rho + \sum n_p = 1/6$$

has no solution in non-negative integers.

Hence

$$M_k(\Gamma) = 0 \quad \text{for } k < 0 \text{ or } k = 2.$$

(3) The possible solns of $(*)$ in non-negative integers for

$k = 4$	is	$n_\rho = 1$	others	are zero
$k = 6$	is	$n_i = 1$	"	"
$k = 8$	"	$n_\rho = 2$	"	"
$k = 10$	"	$n_\rho = 1 = n_i$	"	"

$k = 14$ $\quad n_\rho = 2$, $n_i = 1$, others are zero

Now if f_1, f_2 are 2 non-zero elts of $M_k(\Gamma)$ for $k = 4, 6, 8, 10, 14$, since they have exactly the same zeroes of same order, we can consider the weight zero modular function f_1/f_2

Then f_1/f_2 is actually holom, hence $f_1/f_2 \in M_0(\Gamma) = \mathbb{C}$, and $f_1 = c f_2$ for some constant $c \in \mathbb{C}$. Choose $f_2 = E_k(\tau)$ to get $M_k(\Gamma) = \mathbb{C} E_k$ for $k = 4, 6, 8, 10, 14$.

(4) If $f \in S_k(\Gamma)$ then $n_\infty(f) \geq 1$ and all other terms are non-negative

Hence $S_k = \emptyset$ for $k < 12$ or $k = 14$

Since (*) doesn't have a soln in these cases

If $k = 12$, take $f = \Delta$. (*) implies that the only zero of Δ is at ∞ of order 1

Hence for any $k \geq 12$ any $f \in S_k(\Gamma)$ the modular function f/Δ is actually a modular form of wt $k-12$. ie $f/\Delta \in M_{k-12}(\Gamma)$ and $S_k = \Delta M_{k-12}$ and the map $f \to f/\Delta$ induces an isom $S_k(\Gamma) \cong M_{k-12}(\Gamma)$

5 Finally since E_k does not vanish at ∞ given $f \in M_k(\Gamma)$ with $f = \sum_{n \geq 0} a_n q^n$

$$f - a_0 E_k \in S_k(\Gamma)$$

and hence $M_k(\Gamma) = S_k(\Gamma) + \mathbb{C} E_k$.

As a corollary we have the following dimen. formula

<u>Cor</u> let $k \geq 0$ even. Then

$$\dim M_k = \begin{cases} \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \mod 12 \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \mod 12 \end{cases}$$

$S_k = 0$ if $k < 12$

for $k \geq 12$ $\dim S_k = \begin{cases} \lfloor k/12 \rfloor & k \not\equiv 2 \ (12) \\ \lfloor k/12 \rfloor - 1 & k \equiv 2 \ (12) \end{cases}$

<u>Pf</u> by 5 $\dim M_k = \dim S_k + 1$

by 4 $\dim S_k = \dim M_{k-12}$

Hence $\dim M_k = \dim M_{k-12} + 1$ $(**)$

Now dimension formula holds for $0 \leq k \leq 12$ by parts 1, 2, 3. The two expressions $\lfloor k/12 \rfloor + 1$, $\lfloor k/12 \rfloor$ increase by 1 when k is replaced by $k+12$. Hence the formula holds for $\forall k \geq 0$ by $(**)$.

Next result shows that every $f \in \mathcal{M}_k(\Gamma)$ is a polynomial in E_4, E_6.

<u>Prop 3-6</u> Let $f \in \mathcal{M}_k(\Gamma)$. Then f can be written in the form

$$f(\tau) = \sum_{4i+6j=k} c_{ij} E_4^i E_6^j.$$

<u>Proof</u>. We will use induction on k

For $k = 4, 6, 8, 10, 14$ $\dim \mathcal{M}_k = 1$

and $E_4, E_6, E_4^2, E_6 E_4, E_4^2 E_6$

are in $\mathcal{M}_k$ for $k = 4, 6, 8, 10, 14$ resp.

Hence they must span $\mathcal{M}_k$ in these cases.

Suppose now $k = 12$ or $k \geq 14$ even

recall $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$ is of the above form.

$S_k(\Gamma) = \Delta \mathcal{M}_{k-12}$.

By inductive hyp. any $f \in \mathcal{M}_{k-12}$ is of the above form so the result follows for $S_k(\Gamma)$

To see the result for $\mathcal{M}_k$

Note that for any i, j such that $4i + 6j = k$ we have $E_4^i E_6^j \in \mathcal{M}_k$ (Note such i, j exists since $14 = 4 \cdot 2 + 6 \cdot 1$ and any other even k is either $14 + 4m$ or $16 + 4n$)

$16 = 4 \cdot 1 + 6 \cdot 2$

The constant coef of $E_4^i E_6^j$ is 1.

Now let $f \in M_k$ with $a_0(f) \neq 0$

Then $f - a_0(f) E_4^i E_6^j$ has constant coef 0

hence $f - a_0 E_4^i E_6^j \in S_k$.

But we have already proved the result for S_k hence $f = \sum_{4i+6j=k} c_{ij} E_4^i E_6^j$ ∎

In fact we have

<u>Prop 3.7</u> $\mathcal{M}(\Gamma) = \bigoplus_k \mathcal{M}_k(\Gamma)$
$= \mathbb{C}[E_4, E_6]$

The ring $\mathcal{M}(\Gamma)$ is the free poly. ring with generators E_4, E_6 over $\mathbb{C}$.

<u>Proof</u> (Exercise) Prove that E_4, E_6 are algebraically independent

ie if $P(E_4, E_6) = 0$ for a polynomial $P \in \mathbb{C}[x$ then $P = 0$.

and

<u>Prop 3.8.</u> A basis for $\mathcal{M}_k$ is given by $\{E_4^i E_6^j \mid 4i + 6j = k, \; i, j \in \mathbb{N} \cup \{0\}\}$

<u>Proof</u> We have seen that $\{E_4^i E_6^j \mid 4i + 6j = k\}$ generate $\mathcal{M}_k$

To prove that it forms a basis verify that

$$\#\{(i,j) \in (\mathbb{N} \cup \{0\})^2 \mid 4i + 6j = k\} = \dim M_k.$$

$\boxtimes$

We have a simple but important thm

Thm 3.9. Let $f \in M_k(SL_2(\mathbb{Z}))$ and

$f(z) = \sum_{n=0}^{\infty} a_n q^n$ be its Fourier expansion

If $a_j = 0$ for all $j = 0, \dots \lfloor k/12 \rfloor$ then $f \equiv 0$.

Proof Suppose $f \neq 0$. Since $a_j = 0$ for $j = 1, \dots \lfloor k/12 \rfloor$, $n_\infty(f) > \lfloor \frac{k}{12} \rfloor$

Hence $n_\infty(f)$ is at least $\lfloor \frac{k}{12} \rfloor + 1 > \frac{k}{12}$

but then using the valence formula

$$n_\infty(f) + \sum_{p \neq i, \rho} n_p(f) + \frac{1}{2} n_i(f) + \frac{1}{3} n_\rho(f) = \frac{k}{12}$$

we have the LHS $> k/12 =$ RHS which is a contradiction $\boxtimes$

Cor If $f, g \in M_k(SL_2(\mathbb{Z}))$, $f = \sum a_n q^n$, $g = \sum b_n q^n$ and if $a_j = b_j$ $j = 0, 1, \dots \lfloor k/12 \rfloor$ then $f \equiv g$.